M. QNo → Describe the canonical embedding of a normed linear space E into the second dual space E^{**} and show it is an isometric isomorphism of E into E^{**}. Show that the space R^n is reflexive.

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Or, QNo → Let N be a normed linear space. Then define the dual space N^* and the second dual N^{**}. Prove that the natural embedding is an isometric isomorphism of N into N^*

NOTE:- Or वाला QN. में E तथा E^* के जगह पर N तथा N^* लिखना है।

Proof:- ~~Let $x \in E$ is fixed and $f \in E^*$ is variable,~~

If E is a normed linear space then the dual space E^* is also a normed linear space. We can therefore construct successively the spaces,

$$(E^*)^* = E^{**}, (E^{**})^* = E^{***} \text{ etc.}$$

Each of the spaces $E^*, E^{**}, E^{***}, ----$ etc. are normed linear spaces. The space E^{**} is called the second dual space (or second conjugate space) of E. The space E^{**} is the space of all continuous linear functionals on E^*.

Let $x \in E$ be fixed and $f \in E^*$ be variable clearly for different E^*. We get different values of $f(x)$. Now, for fixed $x \in E$, we define F_x on E^* by setting

$$F_x(f) = f(x) \text{ for all } f \in E^* \quad \text{—} \; ①$$

We show that F_x is a continuous linear functional on E^* . i.e. $F_x \in E^{**}$. We have

$$F_x(\alpha f_1 + \beta f_2) = (\alpha f_1 + \beta f_2)(x) = \alpha f_1(x) + \beta f_2(x)$$
$$= \alpha F_x(f_1) + \beta F_x(f_2) \text{ for all } f_1, f_2 \in E'$$
and all scalars α, β.

Hence, F_x is linear functional on a E^* for each fixed $x \in E$, we have

$$|F_x(f)| = |f(x)| \leq \|f\| \cdot \|x\| \text{ for every } f \in E^*.$$

Therefore, F_x is a Continuous linear function on E^* and as such that $F_x \in E^{**}$.

Hence, we Conclude that for each fixed $x \in E$, there exists a Continuous linear functional $F_x \in E^{**}$ given by ① Hence it is Possible by define $h: E \to E^{**}$ by seting $h(x) = F_x$ for all $x \in E$. ~~It can~~ Then h is a one-to-one linear transformation of E into E^{**} such that

$$\|h(x)\| = \|x\| \text{ for all } x \in E.$$

h is Called the Canonical embedding of E into E^{**}.

For all $x, y \in E$ and all scalars α, β.

$$F_{\alpha x + \beta y}(f) = f(\alpha x + \beta y) - \alpha f(x) + \beta f(y).$$
$$= \alpha F_x(f) + \beta F_y(f).$$
$$= (\alpha F_x + \beta F_y)(f) \quad \forall f \in E^*.$$

$\therefore F_{\alpha x + \beta y} = \alpha F_x + \beta F_y.$

$\therefore h(\alpha x + \beta y) = F_{\alpha x + \beta y} = \alpha F_x + \beta F_y = \alpha h(x) + \beta h(y).$

Therefore h is a linear transformation.

we NOW, Prove that $\|h(x)\| = \|x\|$, we have, $\|h(x)\| = \|F_x\| = \sup\left\{ \frac{|F_x(f)|}{\|f\|} : \|f\| \neq 0 \right\}$

$$= \sup\left\{ \frac{|f(x)|}{\|f\|} : \|f\| \neq 0 \right\}$$

$$= \frac{\|f\| \, \|x\|}{\|f\|} = \|x\| \quad \text{--- (2)}$$

Finally, we now that h is one-one, we have,

$$F_{x-y}(f) = f(x-y) = f(x) - f(y) = F_x(f) - F_y(f)$$
$$= (F_x - F_y)(f).$$

Hence, $F_{x-y} = F_x - F_y$.

$\therefore \|F_x - F_y\| = \|F_{x-y}\| = \|x-y\| \;\forall\; y \in E$.

Hence, if $x \neq y$ i.e. $x - y \neq 0$, we have $\|x-y\| = \|F_x - F_y\| > 0$ i.e. $F_x \neq F_y$, i.e. $h(x) \neq h(y)$

Hence h is a one to one linear map of E into E^{**}. Such that
$$\|h(x)\| = \|x\|.$$

Hence h is an isomorphism between the space E and the set $\{h(x) : x \in E\} \subseteq E^{**}$. This fact is expressed by ~~stating~~ writing, $E \subseteq E^{**}$.

If set of all $\{F_x : x \in E\} = E^{**}$, then h is an isomorphism between E

and E^{**} in this case we say that E is reflexive and we express the fact $E = E^{**}$.

The space R^n is reflexive
$\therefore (R^*)^{**} = ((R^*)^*)^* = (R^*)^* = R^*$.

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Thus, R^n is reflexive.

QN:- Show that the space R^n is reflexive.

or, QN:- The dual space of R^n is isomorphic to R^n.

In short, the dual space of R^n is R^n.

Verification - Let $e_1 = (1, 0, 0, \ldots, 0)$, $e_2 = (0, 1, 0, \ldots, 0)$, $\ldots$ $e_n = (0, 0, 0, \ldots, 1)$ be a basis of R^n. Then any element $x = (\alpha_1, \alpha_2, \ldots, \alpha_n) \in R^n$ can be written as,

$$x = \sum_{k=1}^{n} \alpha_k e_k$$

If f is a continuous linear functional on R^n, then

$$f(x) = f\left(\sum_{k=1}^{n} \alpha_k e_k\right) = \sum_{k=1}^{n} \alpha_k f(e_k) = \sum_{k=1}^{n} \alpha_k \beta_k \quad \text{where } \beta_k = f(e_k)$$

Conversely, every n-tuple $(\beta_1, \ldots, \beta_n) \in R^n$ determines a continuous linear function f of R^n given by $f(x) = \sum_{k=1}^{n} \alpha_k \beta_k$, where $x = (\alpha_1, \ldots, \alpha_n) \in R^n$.

By Cauchy Schwarz inequality, we have

$$|f(x)| \leq \sum_{k=1}^{n} |\alpha_k| |\beta_k| \leq \left(\sum_{k=1}^{n} \alpha_k^2\right)^{\frac{1}{2}} \cdot \left(\sum_{k=1}^{n} \beta_k^2\right)^{\frac{1}{2}}$$

$$= \|x\| \left(\sum_{k=1}^{n} \beta_k^2\right)^{\frac{1}{2}}$$

Hence, f is Continuous and

$$\|f\| = \sup_{x \neq 0} \frac{|f(x)|}{\|x\|} \leq \left(\sum_{k=1}^{n} \beta_k^2 \right)^{\frac{1}{2}}$$

However, if $x = (\beta_1, \ldots \beta_n)$, then

$$f(x) = \sum_{k=1}^{n} \beta_k^2 \text{ and } \frac{|f(x)|}{\|x\|} = \left(\sum_{k=1}^{n} \beta_k^2 \right)^{\frac{1}{2}}$$

$$\therefore \|f\| = \left(\sum_{k=1}^{n} \beta_k^2 \right)^{\frac{1}{2}} = \|y\|$$ where, $y = (\beta_1, \beta_2, \ldots \beta_n)$ $\in R^n$. Hence, the mapping from the dual space $(R^n)^*$ onto R^n defined by

$$f \to y = (\beta_1, \ldots, \beta_n).$$

is norm Preserving. Clearly, the mapping is linear, one-one and onto. Therefore, it is an isomorphism. Hence the dual space or the Conjugate space of R^n is R^n i.e. $(R^n)^* = R^n$.

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~~Hence~~, Thus R^n is reflexive

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